Resampling and Reconstruction with Fractal Interpolation Functions

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***Abstract*— An alternative form of the fractal interpolation function (FIF)—previously unmentioned in the signal processing literature—is noted. This form highlights a simple relationship between fractal and linear interpolation. Using this relationship, many FIF problems can be reduced to a matrix/vector expression. This expression provides a more powerful way to employ the FIF for interpolation and permits its adaptation for reconstruction. Additionally, the alternate form of the FIF allows the construction of fractal functions whose piecewise integrals match observed data.**

***Index Terms*—Fractal interpolation, reconstruction.**

I. Introduction and Review

FRACTAL interpolation, first described in [1], has been used for various data visualization and modeling problems [2]–[7]. In this letter, we describe a form of the FIF that simplifies its implementation and allows it to be used for reconstruction. In this first section, we briefly review the classical form of the FIF as it appears in previous signal processing literature. We then present an alternate form detailed in [1] but largely ignored since. In Section II, we describe how this alternate form leads to a matrix/vector expression for many FIF problems and suggest how this expression can be exploited for reconstruction. In Section III, we describe how the alternate form also permits reconstruction of a continuous function when the observed data represents its piecewise integrals. A simple example of these reconstructions is given in Section IV. In Section V, we make some closing comments.

For uniformly sampled signals, we begin with a data set

$$\{(x_n, y_n) \in D \times \mathbb{R} \colon n \in [0, 1, \cdots, N]\} \tag{1}$$

where $D = [0, 1]$ and $x_n = n/N$. A continuous function $f \colon D \to \mathbb{R}$ is sought that interpolates the data according to

$$f(x_n) = y_n \quad \text{for } n \in [0, 1, \cdots, N]. \tag{2}$$

Following the standard form used in the signal processing literature, FIF's are constructed using N affine mappings of the form

$$w_n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ b_n & \gamma_n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_n \\ d_n \end{pmatrix} \quad \text{for } n = 1, \cdots, N \tag{3}$$

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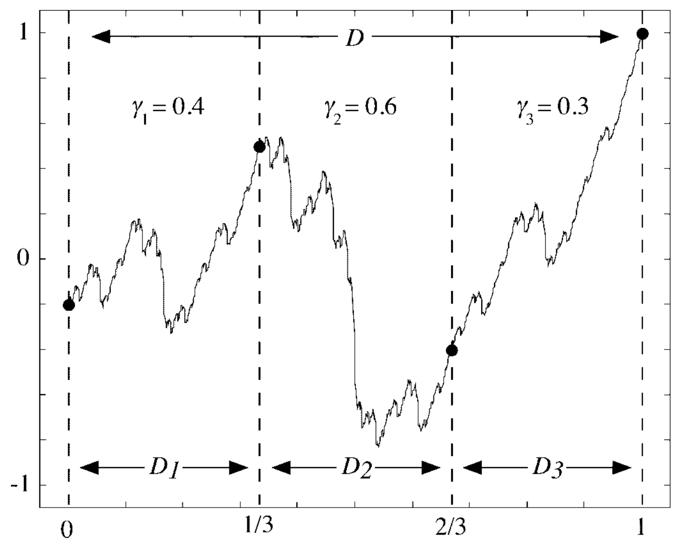

Fig. 1. Example fractal interpolation function with $N = 3$.

with subinterval endpoint constraints

$$w_n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \quad \text{and}$$
$$w_n\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{for } n = 1, \cdots, N. \tag{4}$$

Equations (3) and (4) imply that each map w_n horizontally "shrinks" (by a factor of a_n) and vertically scales (by a factor γ_n) the entire function over the interval D and maps it to the piece of the function over the interval $D_n = [x_{n-1}, x_n]$ (see Fig. 1). For these mappings to be contractive, it is necessary that $|a_n| < 1$ and $|\gamma_n| < 1$. With each γ_n (referred to as *contraction factors*) considered a (fixed) free parameter, $|a_n| < 1$ is guaranteed by the constraints of (4). The contraction factors, however, must be chosen to satisfy $|\gamma_n| < 1$. Under these conditions the collection of mappings defines a hyperbolic iterated function system whose attractor set in $\mathbb{R}^2$ is the graph of a continuous function satisfying (2).

An equivalent form for the FIF associated with (1)–(4) is described in [1]. It can be expressed as

$$w_n(x, y) = (L_n(x), F_n(x, y)) \tag{5a}$$
$$L_n(x) = a_n x + c_n \tag{5b}$$
$$F_n(x, y) = h(L_n(x)) + \gamma_n(f(x) - b(x)) \tag{5c}$$

where the *height function* $h(x)$ is the linear interpolation of the data while the *base function* $b(x)$ is the linear function through (x_0, y_0) and (x_N, y_N). (More general functions $h(x)$ and $b(x)$

are allowed so long as $h(x)$ passes through each of the data points, $b(x)$ passes through the first and last data points and both are continuous.) In this alternate form, the subinterval endpoint constraints become

$$\begin{aligned} L_n(x_0) &= x_{n-1} \quad \text{and} \quad L_n(x_N) = x_n \\ F_n(x_0, y_0) &= y_{n-1} \quad \text{and} \quad F_n(x_N, y_N) = y_n. \end{aligned} \tag{6}$$

Examining (5) we note that $F_n(x,y) = f(L_n(x))$ and hence

$$f(x) = h(x) + \gamma_n[f(L_n^{-1}(x)) - b(L_n^{-1}(x))] \quad \text{for } x \in D_n \tag{7}$$

which is the key expression used in the next sections. In the case of uniform sampling on $D = [0, 1]$ we have $a_n = 1/N$ and (5b) becomes

$$L_n(x) = \frac{1}{N}x + \frac{1}{N}(n-1). \tag{8}$$

II. Reconstruction by Interpolation

Many interpolation and reconstruction problems can be summarized as follows. Given an $(N+1)$-point signal y_n, construct an $(M+1)$-point signal f_n (where $M > N$) that is consistent with y_n according to some model. For basic interpolation, the model states that y_n represents actual points on some continuous function $f(x)$ and f_n is a finer sampling of this function. In reconstruction, the signal y_n often represents the observation of a higher-resolution signal f_n that has been distorted and downsampled. (The effects of noise are not considered in this letter.)

We begin with an $(N+1)$-point signal y_n and seek a signal f_n which has P new points inserted between every sample of y_n. This corresponds to upsampling by a factor of $U = (P+1)$. The interpolated signal f_n will then have $(M+1) = (N+1) + NP$ total points. Although there are various methods [5] to compute points of an FIF, we are interested in only NP specific points. Assuming certain relationships between N and U (to be described shortly), we can use (7) to write

$$f_{nU+k} = h_{nU+k} + \gamma_n(y_{l(k)} - b_{l(k)}) \tag{9a}$$

$$l(k) = \frac{N}{U}k \tag{9b}$$

where $n \in [0, 1, \cdots, N-1]$ and $k \in [0, 1, \cdots, P]$ (which excludes the very last point $f_{M+1} = y_{N+1}$). The $l(k)$ term is derived from the inverse of $L_n(x)$ in (8) and describes how points of the entire signal propagate to points of each subinterval. We have assumed that N and U are such that $l(k)$ is indeed an integer in $[0, 1, \cdots, N]$. This can easily be ensured by signal extension or application of the process to pieces of the signal. The $b_{l(k)}$ term represents $(N+1)$ samples of the base function while h_{nU+k} indicates $(M+1)$ samples of the height function.

We now envision vectors $\boldsymbol{f}, \boldsymbol{h} \in \mathbb{R}^{M+1}$ and $\boldsymbol{\gamma} \in \mathbb{R}^N$. Letting $q_k = y_{l(k)} - b_{l(k)}$ we create a matrix $Q \in \mathbb{R}^{(M+1)\times N}$

$$Q = \begin{pmatrix} \boldsymbol{q} & & & \\ & \boldsymbol{q} & & \\ & & \ddots & \\ & & & \boldsymbol{q} \\ & & & 0 \end{pmatrix} \tag{10}$$

where $\boldsymbol{q} = (0\ q_1\ q_2\ \cdots\ q_P)^T$ so that we can express the FIF problem of (9) simply as

$$\boldsymbol{f} = \boldsymbol{h} + Q\boldsymbol{\gamma}. \tag{11}$$

For interpolation purposes, $\boldsymbol{\gamma}$ is arbitrary so long as $|\gamma_n| < 1$. In many cases, however, we might be interested in choosing $\boldsymbol{\gamma}$ so that $\boldsymbol{f}$ satisfies some properties. For instance, in speech interpolation it might be prudent to make the relationship between $\boldsymbol{f}$ and $\boldsymbol{h}$ as close to linear phase as possible. For signals that are known to exhibit certain fractal properties, we might try to maintain some estimated fractal dimension as in [2]. These are, in essence, reconstruction problems.

A more common reconstruction problem can be stated as follows. Suppose $\boldsymbol{y}$ is a low-resolution observation of a higher-resolution signal $\boldsymbol{z} \in \mathbb{R}^{M+1}$. A known system model implies that $\boldsymbol{y}$ and $\boldsymbol{z}$ are related linearly by $\boldsymbol{y} = \mathrm{A}\boldsymbol{z}$ where $\mathrm{A} \in \mathbb{R}^{(N+1)\times(M+1)}$ represents some (FIR) distortion and downsampling. For this underdetermined problem an estimate $\hat{\boldsymbol{z}}$ is sought so that $\|\boldsymbol{y} - \mathrm{A}\hat{\boldsymbol{z}}\|$ is minimized. Referring to (11) we let $\hat{\boldsymbol{z}} = \boldsymbol{f}$ and let $\boldsymbol{h}$ be some initial estimate of $\boldsymbol{z}$. (The base function implicit in the matrix Q is defined by the first and last points of $\boldsymbol{h}$.) We then seek $\boldsymbol{\gamma}$ to minimize $\|\boldsymbol{y} - \mathrm{A}\boldsymbol{f}\|$. The resulting signal $\boldsymbol{f}$ (an approximation of $\boldsymbol{z}$) is composed of samples of some fractal function. Note that (11) provides a simple expression that can facilitate the selection of $\boldsymbol{\gamma}$. Note also that $h(x)$ is arbitrary in this problem so $\boldsymbol{h}$ can be refined if desired. An example of this approach is given in Section IV.

III. Reconstruction by Integration

In some reconstruction problems, the observed data y_n might represent the piecewise integrals of an unknown continuous function [8], [9]. In other words, $f(x)$ is sought such that $y_n = \int_{D_n} f(x)\,dx$ where now $n \in [1, \cdots, N]$. Applying (7), we can construct such a fractal function. We can then either produce actual points of the resulting function as described previously or integrate over smaller intervals in order to implement resampling.

Simplifying the work in [1] for our purposes, we use (7) to write

$$\begin{aligned} y_n &= \int_{D_n} f(x)\,dx \\ &= \gamma_n \int_{D_n} (f(L_n^{-1}(x)) - b(L_n^{-1}(x)))\,dx \\ &\quad + \int_{D_n} h(x)\,dx. \end{aligned} \tag{12}$$

Letting $\overline{h}_n = \int_{D_n} h(x)\,dx$ and substituting variables yields

$$y_n = \overline{h}_n + \gamma_n a_n \int_D (f(u) - b(u))\,du. \tag{13}$$

Next we let $\overline{F} = \int_D f(u)\, du$ and $\overline{B} = \int_D b(u)\, du$, which gives

$$y_n = \overline{h}_n + a_n(\overline{F} - \overline{B})\gamma_n. \tag{14}$$

We choose $h(x)$ and $b(x)$ so that $h(x)$ is some (again arbitrary) initial guess at $f(x)$ while $b(x)$ must satisfy $b(0) = h(0)$ and $b(1) = h(1)$. Since $\overline{F} = \Sigma_{n=1}^{N} y_n$ and $a_n = 1/N$, only the contraction factor γ_n of (14) is unknown. Since it is required that $|\gamma_n| < 1$, a direct solution of (14) might seem infeasible, perhaps implying a constrained optimization approach is necessary. There is, however, an easier method. First solve (14) for γ_n. If this yields $|\gamma_n| > 1$ then adjust $h(x)$ over D_n so that the solution of (14) does yield $|\gamma_n| < 1$. For example, we might begin with $h(x)$ a piecewise linear function and change it to an appropriate quadratic over D_n if necessary. (This is the approach used in Section IV.) The result is a function $f(x)$ that satisfies $y_n = \int_{D_n} f(x)\, dx$.

To resample $f(x)$ by integration we split each D_n into P equal subintervals D_n^k for $k \in [1, \cdots, P]$ and then compute the NP new integrals of $f(x)$ over these smaller intervals. Assuming N/P is an integer, an expression similar to (14) can be derived:

$$y_n^k = \overline{h}_n^k + a_n(\overline{F}^k - \overline{B}^k)\gamma_n \tag{15}$$

for $n \in [1, \cdots, N]$ and $k \in [1, \cdots, P]$. In (15), y_n^k and $\overline{h}_n^k$ are the integrals of $f(x)$ and $h(x)$ over D_n^k, respectively. Letting D^k represent the splitting of the entire domain D into P subintervals, $\overline{F}^k$ and $\overline{B}^k$ indicate the integrals of $f(x)$ and $b(x)$ over D^k, respectively. Note that $\overline{F}^k$ can be computed by summing the appropriate (and known) piecewise integrals y_n. Note also that it might be desirable to scale the new samples, since they correspond to integration over smaller intervals. This will ensure that they are similar in scale to the original data. An example of this approach is given in the next section.

IV. Examples

We now employ the reconstruction methods of Sections II and III in order to upsample a short segment of a speech signal by a factor of four. The original 257-point signal $\boldsymbol{z}$ was filtered by a five-point FIR filter and then downsampled by a factor of four, yielding the 65-point observation $\boldsymbol{y}$.

For the interpolation-based reconstruction—referring to (11)—we seek $\boldsymbol{\gamma}$ so that $\mathrm{A}\boldsymbol{f} \approx \boldsymbol{y}$ or equivalently

$$\mathrm{AQ}\boldsymbol{\gamma} \approx \boldsymbol{y} = \mathrm{A}\boldsymbol{h}. \tag{16}$$

Since $\boldsymbol{h}$ can be refined, $\boldsymbol{\gamma}$ can easily be found so that (16) is equality. The basic idea is to introduce an artificial contraction factor γ_{N+1} and add an extra column to Q so that the matrix product AQ is invertible. Then (16) can be solved for $\boldsymbol{\gamma}$. The contributions to $\boldsymbol{f}$ from the artificial γ_{N+1} and any contraction factor for which $|\gamma_n| > 1$ are absorbed into $\boldsymbol{h}$. The "INTERPOLATION" reconstruction of Fig. 2 was found in this fashion and therefore has zero reconstruction error—i.e., $\mathrm{A}\boldsymbol{f} = \boldsymbol{y}$.

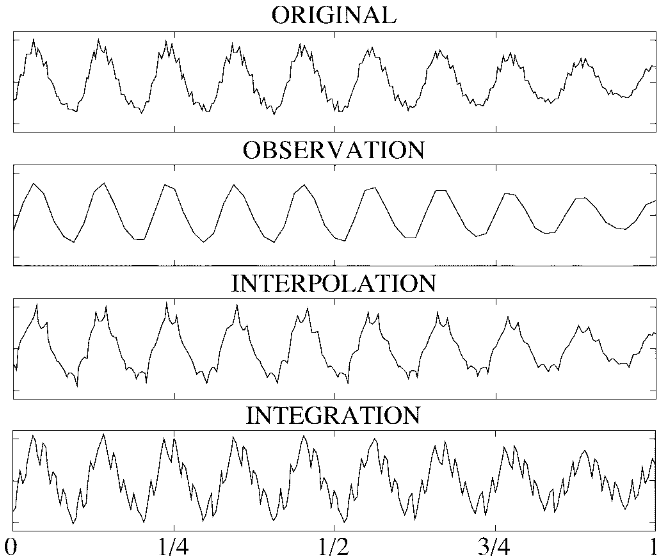

Fig. 2. Fractal reconstruction examples.

For the integration-based reconstruction the first 64 points of $\boldsymbol{y}$ were taken to be the piecewise integrals of some unknown function $f(x)$. This function was found as described in Section III. The reconstruction error here is zero as well since $\int_{D_n} f(x)\, dx = y_n$ by construction. The function was then integrated over quarter-intervals and the result scaled by four to produce the "INTEGRATION" plot shown in Fig. 2.

V. Concluding Remarks

In this letter, we have noted an alternate form of the FIF and shown how it can be applied to reconstruction problems. Although only modest examples were given, the simplicity of the approaches described should permit a wide variety of fractal reconstructions.

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